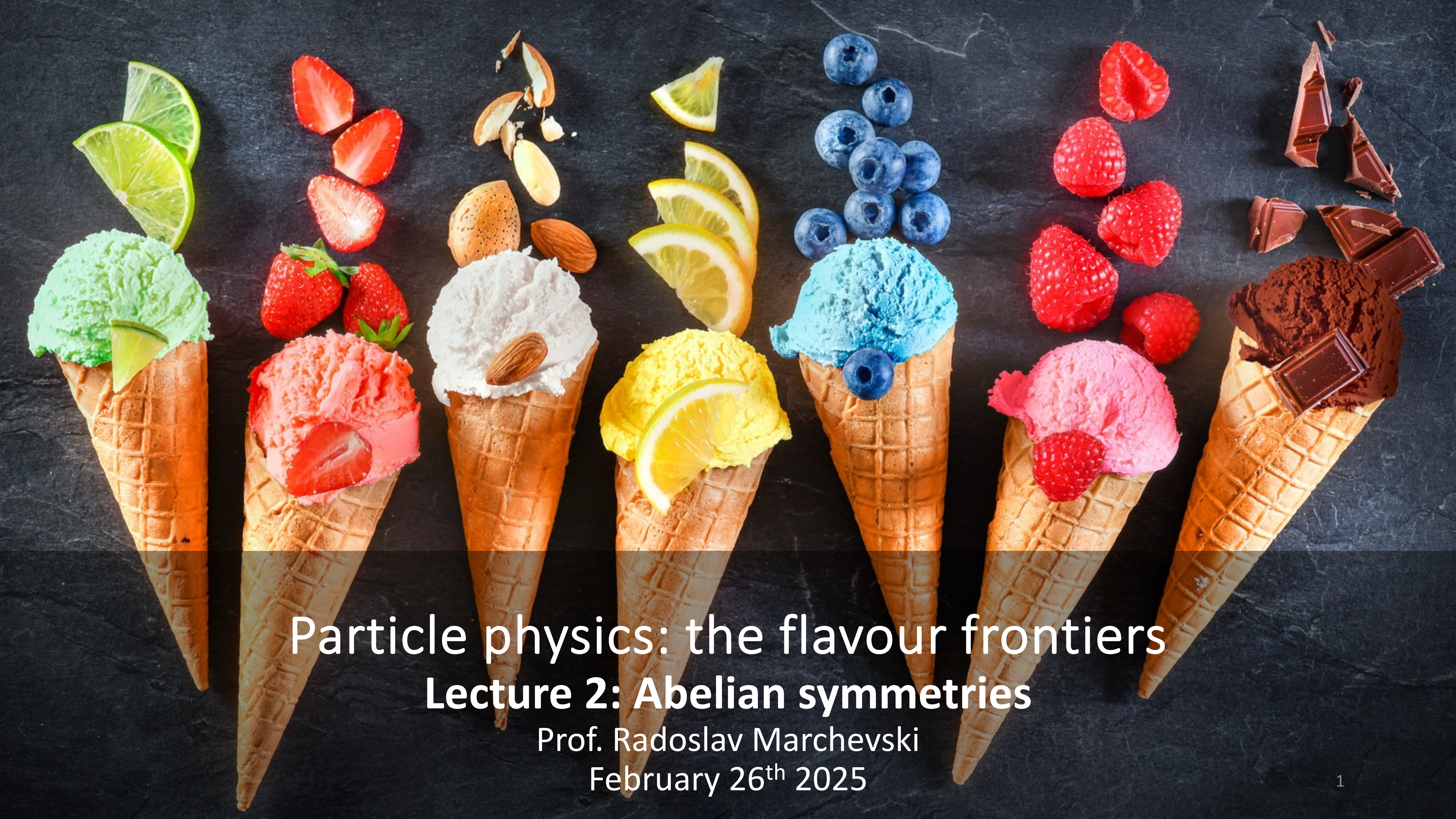

Particle physics: the flavour frontiers

Lecture 2: Abelian symmetries

Prof. Radoslav Marchevski

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Short recap

<u>Last time we discussed</u>

- what patterns do we see by looking at flavour processes in data
  - lepton universality, suppressed flavour-changing neutral currents, generation heirarchy
- how to construct the most general Lagrangians of Nature with scalar and fermion fields
- symmetry and its relation to observable phenomena (conservation laws)

Today's learning targets

<u>Today you will …</u>

- get familiar with different types of Abelian internal symmetries
- learn what happens if we impose these symmetries on the Lagrangian and what is the allowed spectrum
- see an example of an Abelian theory: Quantum Electrodynamics (QED)

Symmetries

“ My methods [of algebra] are really methods of working and thinking; this is why they have crept in everywhere anonymously. ”

Emmy Noether

- Symmetries are essential for understanding flavour physics!
- We will discuss various types of internal symmetries
- We will introduce the notion of charge and its relation to symmetries

Symmetries

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Abelian ⇔ Non-Abelian

Discrete ⇔ Continuous

Global ⇔ Local

Chiral ⇔ Vectorial

Global discrete symmetries

- **<u>Global symmetry:</u>** *a symmetry under transformations that are constant in spacetime*
- Let's start with a simple example of imposing an internal global discrete symmetry on our Lagrangian of a real scalar field ϕ

$$\mathcal{L}_S = \frac{1}{2}\left(\partial_\mu \phi\right)(\partial^\mu \phi) \; - \frac{m^2}{2}\phi^2 - \frac{\eta}{2\sqrt{2}}\phi^3 - \frac{\lambda}{4}\phi^4$$

- We impose a discrete symmetry: we demand that $\mathcal{L}_S(\phi) = \mathcal{L}_S(-\phi)$
- Which terms obey the symmetry, and which don't?
- Which conservation law corresponds to this symmetry?

Global discrete symmetries

$$\mathcal{L}_S = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4}\phi^4$$

- By imposing the symmetry under $\phi \to -\phi$ transformation, we force $\eta = 0$
- ϕ-parity $= (-1)^n$
- The number of ϕ particles described by $\mathcal{L}_S$ can change only by an *even number:* ϕ-parity conservation
- In the language of group theory: Z_2 group with two elements – even $(+)$ and odd $(-)$
  - $\mathcal{L}_S$ belongs to the even representation of Z_2
  - ϕ belongs to the odd representation of Z_2

Z_2	$(+)$	$(-)$
$(+)$	$(+)$	$(-)$
$(-)$	$(-)$	$(+)$

ϕ^{2n} – even representation
ϕ^{2n+1} – odd representation

Global continuous symmetries

- Extend our discussion to symmetries under rotation **in some internal space**
- Some fields not invariant under rotations but the combinations that appear in the Lagrangian are
- *Example:* complex scalar field ϕ (ϕ_R and ϕ_I real scalar fields)

$$\phi = \frac{1}{\sqrt{2}}(\phi_R + i\phi_I)$$

- The most general renormalizable $\mathcal{L}(\phi_R, \phi_I)$ is given by

$$\mathcal{L}(\phi_R, \phi_I) = \frac{1}{2}\delta_{ij}(\partial_\mu \phi_i)(\partial^\mu \phi_j) - \frac{m_{ij}^2}{2}\phi_i\phi_j - \frac{\eta_{ijk}}{6}\phi_i\phi_j\phi_k - \frac{\lambda_{ijkl}}{24}\phi_i\phi_j\phi_k\phi_l, \qquad i,j,k,l = R,I$$

- Consider rotations in the complex plane: $\begin{pmatrix}\phi_R \\ \phi_I\end{pmatrix} \to O\begin{pmatrix}\phi_R \\ \phi_I\end{pmatrix}, \qquad O = \begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}$

Which group is that?

Global continuous symmetries

- Global symmetry $\Longrightarrow \theta$ does not depend on x_μ
- Imposing a global $SO(2)$ symmetry on $\mathcal{L}(\phi_R, \phi_I)$ forbids many terms and relates others

$$\mathcal{L}(\phi_R, \phi_I) = \frac{1}{2}\delta_{ij}(\partial_\mu \phi_i)(\partial^\mu \phi_j) - \frac{m_{ij}^2}{2}\phi_i\phi_j - \frac{\eta_{ijk}}{6}\phi_i\phi_j\phi_k - \frac{\lambda_{ijkl}}{24}\phi_i\phi_j\phi_k\phi_l, \qquad i,j,k,l = R, I$$

$SO(2)$ symmetry

$$\begin{pmatrix} \phi_R \\ \phi_I \end{pmatrix} \to O \begin{pmatrix} \phi_R \\ \phi_I \end{pmatrix}, \qquad O = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\mathcal{L}(\phi_R, \phi_I) = \frac{1}{2}(\partial_\mu \phi_R)(\partial^\mu \phi_R) + \frac{1}{2}(\partial_\mu \phi_I)(\partial^\mu \phi_I) - \frac{m^2}{2}(\phi_R\phi_R + \phi_I\phi_I) - \frac{\lambda}{4}(\phi_R^4 + \phi_I^4 + 2\phi_R^2\phi_I^2)$$

Global continuous symmetries

- Alternatively, we can formulate the transformation law directly in terms of the complex field ϕ

$$\phi \to e^{i\theta}\phi, \qquad \phi^\dagger \to e^{-i\theta}\phi^\dagger$$

- Imposing this symmetry on $\mathcal{L}(\phi, \phi^\dagger)$ leads to

$$\mathcal{L}(\phi, \phi^\dagger) = (\partial^\mu \phi^\dagger)\,(\partial_\mu \phi) - m^2 \phi^\dagger \phi - \lambda(\phi^\dagger \phi)^2$$

- Rotations in a one-dimensional complex plane: $U(1)$ symmetry (mathematically equivalent to $SO(2)$)
  - the different names reflect the way we think about the underlying space

- **<u>Important points:</u>**
  - the three terms do not violate any internal symmetry (no way to forbid them by imposing internal symmetry!)
  - the same result can be obtained by scaling θ by an any non-zero number: $\phi \to e^{iq\theta}\phi$
  - different situation if we have more than one complex field

Charge (global symmetry)

- **<u>Charge from electromagnetism:</u>**

  i. *sets the strength of the interaction of the fermions with the photon*

  ii. *it is a conserved quantity*

- **Internal continuous symmetries ⟺ conserved charges (Noether's theorem)**

- Theory with two complex scalar fields, ϕ_1 and ϕ_2

$$\phi_1 \to e^{iq_1\theta}\phi_1, \qquad \phi_2 \to e^{iq_2\theta}\phi_2 \qquad\qquad \phi_1^\dagger \to e^{-iq_1\theta}\phi_1^\dagger, \qquad \phi_2^\dagger \to e^{-iq_2\theta}\phi_2^\dagger$$

- $q_i(-q_i)$ is the charge of the $\phi_i\left(\phi_i^\dagger\right)$ field

- We can't set $q_1 = q_2 = 1$ because q_1/q_2 is a physical quantity

Charge (global symmetry)

- Theory with two complex fields, ϕ_1 and ϕ_2, and charges $q_1 = 1$ and $q_2 = 3$

$$\phi_1 \to e^{iq_1\theta}\phi_1, \qquad \phi_2 \to e^{iq_2\theta}\phi_2 \qquad\qquad \phi_1^\dagger \to e^{-iq_1\theta}\phi_1^\dagger, \qquad \phi_2^\dagger \to e^{-iq_2\theta}\phi_2^\dagger$$

- The most general renormalisable Lagrangian is

$$\mathcal{L} = \left(\partial^\mu \phi_1^\dagger\right)\left(\partial_\mu \phi_1\right) + \left(\partial^\mu \phi_2^\dagger\right)\left(\partial_\mu \phi_2\right) - m_1^2\phi_1^\dagger\phi_1 - m_2^2\phi_2^\dagger\phi_2 - \lambda_{11}\left(\phi_1^\dagger\phi_1\right)^2 - \lambda_{22}\left(\phi_2^\dagger\phi_2\right)^2$$
$$- \lambda_{12}\left(\phi_1^\dagger\phi_1\right)\left(\phi_2^\dagger\phi_2\right) - \left(\eta\phi_1^3\phi_2^\dagger + \mathrm{h.c.}\right)$$

- <u>Comments</u>
  - for a term to be allowed, the sum of the charges of the fields in this term must be zero
  - all interactions allowed by the symmetry conserve the charge (each term in $\mathcal{L}$ carries an overall charge zero)
  - charge is related to a phase shift and can only be assigned to complex fields ($q = 0$ for real fields)
  - two terms often used in the physics jargon instead of *charges*: *quantum numbers (*QNs*)* and *representations*

Symmetries and fermion masses

- Define a $U(1)$ phase transformation of Weyl fermions:

$$\psi_L \to e^{iq_L\theta}\psi_L, \qquad \psi_R \to e^{iq_R\theta}\psi_R \qquad\qquad \overline{\psi_L} \to e^{-iq_L\theta}\overline{\psi_L}, \qquad \overline{\psi_R} \to e^{-iq_R\theta}\overline{\psi_R}$$

- *Note:* ψ_R and $\overline{\psi_R^c}$ transform in the same way under all symmetries

- Consider a theory with a single left-handed and single right-handed fermion fields + $U(1)$ symmetry
  - the symmetry is called *chiral* if $q_L \neq q_R$
  - the symmetry is called *vectorial* if $q_L = q_R$
- Generalising to the case with several fermion fields, the symmetry is *vectorial* if all LH and RH fields can be matched into pairs with the same charge $q_{Li} = q_{Ri}$ for each i, and *chiral* otherwise
- Important result: any chiral theory violates C and P (sufficient but not necessary condition)

Symmetries and fermion masses

$$\mathcal{L}_m = \frac{m_{MR}}{2}\overline{\psi_R^c}\psi_R + \frac{m_{MR}}{2}\overline{\psi_L^c}\psi_L + m_D\overline{\psi_L}\psi_R + h.c.$$

- To allow Dirac mass terms, the charges of $\overline{\psi_L}$ and ψ_R must be opposite (true when $q(\psi_L) = q(\psi_R)$)
- To allow Majorana mass terms, a fermion field must be neutral under all $U(1)$ symmetries
- Consider a theory with m left-handed (LH) and n right-handed (RH) fields
- *Case I:* all $(m+n)$ fields carry the same charges, $q \neq 0$
  - Majorana mass terms vanish
  - Dirac mass terms for an $m \times n$ general complex matrix m_D

$$\mathcal{L}_m = (m_D)_{ij}(\overline{\psi_L})_i(\psi_R)_j + h.c.$$

  - if $m \leq n \Longrightarrow m$ Dirac fermions and $(n-m)$ massless RH fermions
  - if $n \leq m \Longrightarrow n$ Dirac fermions and $(m-n)$ massless LH fermions
  - in the SM the charged fermions are present in three copies ($m = n = 3$) with the same QN→ $m_D^{(3\times3)}$

Symmetries and fermion masses

$$\mathcal{L}_m = \frac{m_{MR}}{2}\overline{\psi_R^c}\psi_R + \frac{m_{MR}}{2}\overline{\psi_L^c}\psi_L + m_D\overline{\psi_L}\psi_R + h.c.$$

- Consider a theory with m left-handed (LH) and n right-handed (RH) fields
- *Case II:* all $(m+n)$ fields are neutral, $q = 0$
  - both Majorana and Dirac mass terms are allowed
  - $(m+n)\times(m+n)$ symmetric complex matrix M_ψ

$$\begin{pmatrix} \overline{\psi_L} & \overline{\psi_R^c} \end{pmatrix} \begin{pmatrix} m_{ML}^{(m\times m)} & m_D^{(m\times n)} \\ m_D^{T(n\times m)} & m_{MR}^{(n\times n)} \end{pmatrix} \begin{pmatrix} \psi_L^c \\ \psi_R \end{pmatrix}$$

  - the $(m+n)$ mass eigenstates are Majorana fermions
  - in the SM, neutrinos are the only neutral fermions with $m = 3$ and $n = 0$
  - if they have Majorana masses then their mass matrix is $m_{ML}^{(3\times 3)}$

Symmetries and fermion masses

	Dirac	Majorana
# of degrees of freedom	4	2
Representation	vector	neutral
Mass matrix	$m \times n$, general	$(m+n) \times (m+n)$, symmetric
SM fermions	quarks, charged leptons	neutrinos (?)

- Important differences between Majorana and Dirac particles
- Lesson to be extracted from these differences: charged fermions in a chiral representation ($q_L \neq q_R$) are massless $\Longrightarrow$ if we encounter massless fermions in nature, we can explain their masslessness from symmetry principles

Local (gauge) symmetries

- **<u>Local (gauge) symmetries:</u>** symmetries, where the transformation can be different at different space-time points $\theta(x_\mu)$
- Local symmetries have far-reaching consequences!
- All internal symmetries imposed in defining the SM are local
- Local transformation of a complex scalar field:

$$\phi(x) \to e^{iq\theta(x)}\phi(x), \qquad \phi^\dagger(x) \to e^{-iq\theta(x)}\phi^\dagger(x)$$

- All terms in the Lagrangian that do not involve derivatives of fields and are invariant under a global symmetry are also invariant under the corresponding local symmetry

Local (gauge) symmetries

- The derivative terms are not invariant under the local symmetry

$$\partial^\mu \phi(x) \to \partial^\mu\left[e^{iq\theta(x)}\phi(x)\right] = e^{iq\theta(x)}\phi(x) + iqe^{iq\theta(x)}[\partial^\mu\theta(x)]\phi(x)$$

- $\Longrightarrow$ the kinetic term of a scalar is not invariant under the local symmetry (same is true for fermion fields)

$$\partial^\mu \phi^\dagger(x)\partial_\mu\phi(x) \to \left[\partial^\mu\phi^\dagger(x) - iq[\partial^\mu\theta(x)]\phi^\dagger(x)\right]\left[\partial_\mu\phi(x) + iq\left[\partial_\mu\theta(x)\right]\phi(x)\right] \neq \partial^\mu\phi^\dagger(x)\partial_\mu\phi(x)$$

- **Consequence:** in a theory with only scalar and fermions fields a local symmetry acting on these fields forbids the kinetic terms.
  - this is a BIG deal: such fields are not dynamic and can't describe the particles we observe in nature!
  - we have to find a way to "correct" for this

Local (gauge) symmetries

- Replace the kinetic term $\partial^\mu \phi$ with a so-called "covariant" derivative $D^\mu \phi$
- $D^\mu \phi$ should transform under the local symmetry as $\partial^\mu \phi$ under the global symmetry

$$D^\mu \phi \to e^{iq\theta} D^\mu \phi$$

- Using the local transformation $\phi \to e^{iq\theta}\phi$ (x-dependence implicit) we get for the covariant derivative

$$D^\mu = \partial^\mu + igqA^\mu \qquad\qquad A^\mu \to A^\mu - \frac{1}{g}\partial^\mu \theta$$

- g is a dimensionless constant which we take to be positive (coupling constant)
- $A^\mu(x)$ is a vector field that transforms under the local symmetry (gauge field)
- Following our principles we must add a kinetic term for the gauge field A^μ

Local (gauge) symmetries

- To add a kinetic term for A^μ we define the field strength tensor $F^{\mu\nu}$

$$[D^\mu, D^\nu] = igqF^{\mu\nu} \qquad\qquad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

- The Lorentz-invariant kinetic term for the gauge field is (invariant under the local symmetry)

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

- **Important points:**
  - given the kinetic term, A^μ is a dynamic field and its excitations are physical particles (e.g. the photon)
  - a mass term of the form $\frac{1}{2}m^2 A^\mu A_\mu$ is not allowed (show that it breaks gauge invariance)→ gauge bosons have only two degrees of freedom
  - a gauge field related to an Abelian symmetry does not couple to itself
  - transformation law for the gauge field is additive (A^μ transforms like a phase)
  - if a local symmetry decomposes $U(1)_a \times U(1)_b$ → two gauge fields A^a_μ, A^b_μ + two independent couplings g_a, g_b

Charge (local symmetry)

- Global symmetry implies charge conservation
- In the case of local symmetry there is an additional implication to the charge

Any ideas what is the implication?

Charge (local symmetry)

- Global symmetry implies charge conservation
- In the case of local symmetry there is an additional implication to the charge: it sets the strength of the interaction with the gauge boson
- Consider a local $U(1)$ symmetry. The covariant derivative of a field with charge q is

$$D^{\mu} = \partial^{\mu} + igqA^{\mu}$$

- The kinetic term of the fermion field is

Interaction between the fermion and the vector field!

$$i\bar{\psi}\gamma^{\mu}D_{\mu}\psi = i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - gq\bar{\psi}\gamma^{\mu}A_{\mu}\psi$$

- The strength of the interaction is governed by the coupling constant g times the charge q

Charge (local symmetry)

Interaction between the fermion and the vector field!

$$D^\mu = \partial^\mu + igqA^\mu \qquad i\bar{\psi}\gamma^\mu D_\mu\psi = i\bar{\psi}\gamma^\mu\partial_\mu\psi - gq\bar{\psi}\gamma^\mu A_\mu\psi$$

- The larger the charge q, the stronger the coupling to the gauge boson
- Comments:
  - D^μ depends on the charge of the field on which it acts (different for fields with different charges)
  - what appears in the Lagrangian is gq, not g and q separately
  - for Abelian local symmetries one can rescale the coupling and the charge to keep gq constant
  - when we have several fields with the same charges one should rescale all of them the same way
  - ratio between the charges of all fields (e.g. q_2/q_1) is physical and sets the relative strength of the interaction of ψ_2 and ψ_1 with the gauge field A_μ (different in the non-Abelian case)

Abelian symmetries summary

$$\mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\psi} + \mathcal{L}_{\phi} + \mathcal{L}_{\text{Yuk}}$$

- $\mathcal{L}_{\text{kin}}$: free propagation through spacetime of all dynamic fields
- $\mathcal{L}_{\psi}$: Fermion mass terms
- $\mathcal{L}_{\phi}$: scalar potential
- $\mathcal{L}_{\text{Yuk}}$: Yukawa interactions between the scalar and fermion fields

- The most general renormalisable $\mathcal{L}$ with scalar, fermion, and gauge can be decomposed into four terms
- The invariance under local $U(1)$ symmetry requires the introduction of vector (gauge) bosons
- The vector boson interacts with all scalars and fermions charged under the symmetry
- Global symmetry $\Longrightarrow$ charge conservation
- Local symmetry $\Longrightarrow$ the charge sets the strength of the interaction with the gauge field
- In the SM we impose only local symmetries (Abelian and non-Abelian)

Example of an Abelian theory: Quantum electrodynamics (QED)

- Simple version of QED with two Dirac fermions:
  - local $U(1)_{\mathrm{EM}}$ theory
  - two Dirac fermions: $\ell_L^i(-1), \ell_R^i(-1) \quad [i = 1, 2]$
  - no scalar fields $\mathcal{L}_{\mathrm{Yuk}} = \mathcal{L}_\phi = 0$

$$\mathcal{L}_{\mathrm{kin}} = i\overline{\ell_L^1}\gamma^\mu D_\mu \ell_L^1 + i\overline{\ell_L^2}\gamma^\mu D_\mu \ell_L^2 + i\overline{\ell_R^1}\gamma^\mu D_\mu \ell_R^1 + i\overline{\ell_R^2}\gamma^\mu D_\mu \ell_R^2$$

$$-\mathcal{L}_\psi = \overline{\ell_L^i} m_{ij} \ell_R^j + \mathrm{h.c.} = \left(\overline{\ell_L^1}, \overline{\ell_L^2}\right) \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{pmatrix} \overline{\ell_R^1} \\ \overline{\ell_R^2} \end{pmatrix} + \mathrm{h.c.}$$

V_L, V_R – unitary matrices

$$V_L \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} V_R^\dagger = \begin{bmatrix} m_e & 0 \\ 0 & m_\mu \end{bmatrix} \Longrightarrow -\mathcal{L}_\psi = \overline{\ell_L^i} V_L^\dagger V_L m_{ij} V_R^\dagger V_R \ell_R^j + \mathrm{h.c.}$$

Diagonalise the mass matrix
(question 3.5 in the book)

Example of an Abelian theory: Quantum electrodynamics (QED)

- Rotation from the original basis to the new one where the mass matrix is diagonal (mass basis)

$$\begin{pmatrix} \ell_L^1 \\ \ell_L^2 \end{pmatrix} \to \begin{pmatrix} e_L \\ \mu_L \end{pmatrix} = V_L \begin{pmatrix} \ell_L^1 \\ \ell_L^2 \end{pmatrix}, \qquad \begin{pmatrix} \ell_R^1 \\ \ell_R^2 \end{pmatrix} \to \begin{pmatrix} e_R \\ \mu_R \end{pmatrix} = V_R \begin{pmatrix} \ell_R^1 \\ \ell_R^2 \end{pmatrix}$$

In the new basis

$$-\mathcal{L}_\psi = m_e \bar{e} e + m_\mu \bar{\mu} \mu$$

- The rotation keeps the kinetic term invariant, the full QED Lagrangian is then

$$\mathcal{L}_{\text{kin}} = i\bar{e}\gamma^\mu D_\mu e + i\bar{\mu}\gamma^\mu D_\mu \mu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - m_e \bar{e} e - m_\mu \bar{\mu} \mu$$

- Three free parameters: $\alpha = e^2/4\pi, m_e, m_\mu$

$$\boldsymbol{\alpha^{-1} = 137.035999084(21), \quad m_e = 0.51099895000(15)\ \mathrm{MeV}/c^2, \quad m_\mu = 105.6583755(23)\ \mathrm{MeV}/c^2}$$

Summary of Lecture 2

<u>Main learning outcomes</u>

- Get familiar with different types of Abelian internal symmetries
- Learn what happens if we impose these symmetries on the Lagrangian and what is the spectrum of allowed fields and interactions
- See an example of an Abelian theory: Quantum Electrodynamics (QED)